The Lee Fields Medal VI: SOLUTIONS

Time Allowed: Up to Two Hours and 15 Minutes

Tables and calculators may be used. Each question worth 10 Marks.

1. If N has no prime factors $p \leq \sqrt{N}$, then N is prime. True or false. Justify your answer.

Solution: (*I was expecting something like:*) Let $N = p_1 \cdot p_2 \cdots p_k$ be the prime decomposition of N. By assumption, each $p_i > \sqrt{N}$, and so we have, for $k \geq 2$:

$$N > \sqrt{N} \cdot \sqrt{N} \cdots \sqrt{N}^k = N(\sqrt{N})^{k-2} > N,$$

which is impossible, and so $k = 1$, $N = p_1$, a prime.

Your winner, Alan MacDonald produced a nice variation of this; using the *contrapositive*:

If N is not prime, then N has a prime factor $p \leq \sqrt{N}$.

Assume that N is not prime, then it has a factor a and:

$$N = a \cdot \frac{N}{a},$$

with $N/a \in \mathbb{N}$. Note every factor of a is a factor of N. If $a \leq \sqrt{N}$ then we are done because a has a prime factor $p \leq a \leq \sqrt{N}$. Assume that $a > \sqrt{N}$. Then:

$$\begin{aligned} \sqrt{N} &< a \\ \underset{\times\sqrt{N}}{\times} \quad N &< a\sqrt{N} \\ \underset{\div a}{\Longrightarrow} \quad \frac{N}{a} &< \sqrt{N} \end{aligned}$$

That is the second factor of $N = a \cdot N/a$ must be less than $\sqrt{N}$ and N/a must have a prime factor $p \leq N/a \leq \sqrt{N}$, which is also a factor of N.

Unfortunately, and this was an unintended catch, both of these proofs have flaws. The flaw being that it is *not* true that every natural number N has a prime factor! The number one does not have a prime factor!

So the catch is that the statement is actually false, because $N = 1$ has no prime factors less than $\sqrt{N} = 1$ but neither is one prime!

2. Where $0° \leq \theta < 360°$, show that if $z \in \mathbb{C}$ satisfies

$$z + \frac{1}{z} = 2\cos(\theta),$$

then its modulus $|z| = 1$.

Solution: If $z = 0$, $1/z$ is undefined and the equation cannot hold. Therefore assume that $z \neq 0$ and multiply both sides by z and arrange into a quadratic equation:

$$\begin{aligned} \underset{\times z}{\Longrightarrow} \quad & z^2 + 1 = 2\cos(\theta)z \\ \underset{-2\cos(\theta)z}{\Longrightarrow} \quad & z^2 - 2\cos(\theta)z + 1 = 0 \end{aligned}$$

Using the "$-b$"-formula:

$$\begin{aligned} z &= \frac{2\cos(\theta) \pm \sqrt{(-2\cos(\theta))^2 - 4(1)(1)}}{2(1)} \\ &= \frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2} \\ &= \frac{2\cos(\theta) \pm \sqrt{4(\cos^2(\theta) - 1)}}{2} \\ &= \frac{2\cos(\theta) \pm \sqrt{4}\sqrt{-\sin^2(\theta)}}{2} \\ &= \frac{2\cos(\theta) \pm 2i\sin(\theta)}{2} \\ &= \cos(\theta) \pm i\sin(\theta). \end{aligned}$$

Now we calculate the modulus:

$$|z| = \sqrt{\cos^2(\theta) + (\pm\sin(\theta))^2} = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1.$$

3. Solve, for $0 \leq \theta \leq 90°$,

$$2^{\sin^2\theta} + 2^{\cos^2\theta} = 3.$$

Solution: Let $t = \sin^2\theta$ and note that $t + \cos^2\theta = 1 \implies \cos^2\theta = 1 - t$. Now the equation reads:

$$\begin{aligned} 2^t + 2^{1-t} &= 3 \\ \implies 2^t + \frac{2^1}{2^t} &= 3 \end{aligned}$$

Let $y = 2^t$:

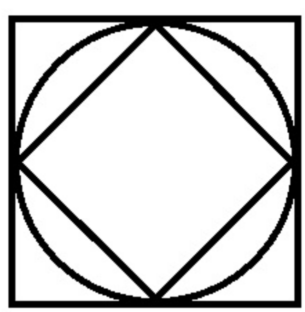
$$\begin{aligned} y + \frac{2}{y} &= 3 \\ \underset{\times y}{\implies} y^2 + 2 &= 3y \\ \underset{-3y}{\implies} y^2 - 3y + 2 &= 0 \\ \implies (y-1)(y-2) &= 0 \\ \implies y &= 1, 2. \end{aligned}$$

This gives $2^t = 1$ or $2^t = 2$ that is $t = 0$ or $t = 1$ that is $\sin^2\theta = 0$ or $\sin^2\theta = 1$. In the range $0 \leq \theta \leq 90°$, $\sin\theta \geq 0$ so we can take square roots to find $\sin\theta = 0$ or $\sin\theta = 1$, and applying $\sin^{-1}$ we find $\theta = 0$ or $\theta = 90°$.

4. Use a circle of diameter one to show that

$$\frac{4}{\sqrt{2}} < \pi < 4.$$

Solution: The solution boils down to drawing this picture:

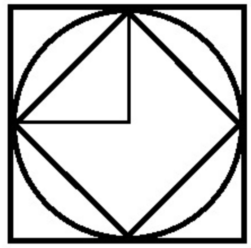


Let $\mathcal{S}$ be the 'outside' square, $\mathcal{C}$ the circle, and $\mathcal{D}$ the 'inside' diamond/square. We have, where P is perimeter, that:

$$P_{\mathcal{D}} < P_{\mathcal{C}} < P_{\mathcal{S}}.$$

Recalling the diameter is one, the second two of these are easily seen to be π and 4.

For the other perimeter, make a right-angled triangle with side lengths $r = 1/2$ and hypotenuse $h := P_{\mathcal{D}}/4$.



Using Pythagoras:

$$\begin{aligned} & h^2 = r^2 + r^2 \\ \implies & h^2 = \frac{1}{2} \\ \underset{\sqrt{\,}, h>0}{\implies} & h = \frac{1}{\sqrt{2}} \\ \implies & P_{\mathcal{D}} = 4h = \frac{4}{\sqrt{2}}. \end{aligned}$$

Note that thankfully the question says *show* rather than *prove*. How to *prove* that

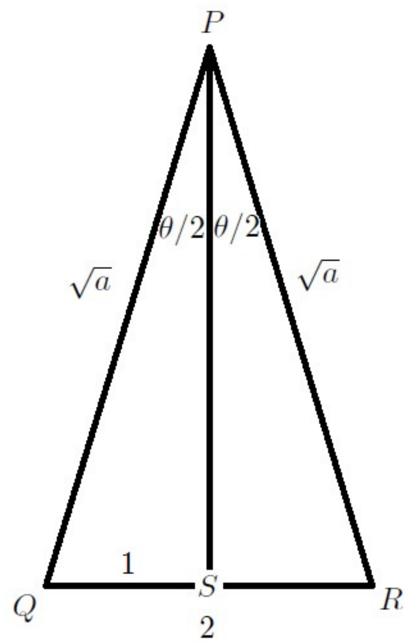
$$P_{\mathcal{C}} < P_{\mathcal{S}}?$$

I am not sure. We can instead argue using areas if we want to avoid this difficulty.

5. Use an isosceles triangle with side-lengths $\sqrt{a}$, $\sqrt{a}$, 2 to prove that for all $a > 1$,

$$2\cos^{-1}\sqrt{\frac{a-1}{a}} = \cos^{-1}\left(\frac{a-2}{a}\right).$$

Solution: Starting with a picture, after dropping a perpendicular onto $[QR]$ and noting that where $\theta = \angle QPR$ this bisects 0:



Writing down the Cosine Rule in ΔPQR we find:

$$\begin{aligned}
(2)^2 &= (\sqrt{a})^2 + (\sqrt{a})^2 - 2\sqrt{a}\sqrt{a}\cos(\theta) \\
\Longrightarrow 4 &= 2a - 2a\cos\theta \\
\underset{+2a\cos\theta-4}{\Longrightarrow} 2a\cos\theta &= 2a - 4 \\
\underset{\div 2a}{\Longrightarrow} \cos\theta &= \frac{2a-4}{2a} \\
\underset{\times(1/2)/(1/2)}{\Longrightarrow} \cos\theta &= \frac{a-2}{a} \\
\underset{\cos^{-1}}{\Longrightarrow} \theta &= \cos^{-1}\left(\frac{a-2}{a}\right)
\end{aligned}$$

Now, consider the right-angled ΔPQS:

$$(\sqrt{a})^2 = |QS|^2 + |PS|^2$$
$$\implies a = 1 + |PS|^2$$
$$\underset{\sqrt{\circ - 1}}{\implies} |PS| = \sqrt{a-1}$$
$$\implies \cos(\theta/2) = \frac{\sqrt{a-1}}{\sqrt{a}}$$
$$\implies \cos(\theta/2) = \sqrt{\frac{a-1}{a}}$$
$$\underset{\times 2 \circ \cos^{-1}}{\implies} \theta = 2\sqrt{\frac{a-1}{a}}.$$

6. Alice rolls one fair six-sided die. She takes the result of the roll, divides it by 6, and then takes the square root. This is her score.

   Bob rolls two fair six-sided dice. He takes the higher of the two rolls and divides it by 6. This is his score.

   Who do we expect to have a higher score? Justify your answer.

   Solution: This question requires some interpretation... *who do we expect to have a higher score?* Where X is Alice's score and Y is Bob's score, there are two interpretations:

   (a) who is expected to have a higher average score;

   (b) what is the probability that one score is higher than the other.

   The first of these is $\mathbb{E}[X]$ vs $\mathbb{E}[Y]$, and the second is $\mathbb{P}[X > Y]$ vs $\mathbb{P}[Y > X]$... and in general these are not the same thing! In general, games where one score is occasionally a lot higher than the other can see this. For example, consider a biased coin such that:

$$\mathbb{P}[H] = \frac{1}{3} \text{ and } \mathbb{P}[T] = \frac{2}{3}.$$

   Let X be a score, equal to 8 if the biased coin is H, and 1 if the biased coin is T. Now let Y be a score, equal to two if a fair coin is H, and three if the fair coin is T. Now,

$$\mathbb{E}[X] = \sum X_i \mathbb{P}[X = X_i] = 8 \times \frac{1}{3} + 1 \times \frac{1}{3} = 3,$$

   while $\mathbb{E}[Y] = 2.5$, so in the first sense X is expected to be higher.

   However, where the game is to have a higher score, we can consider $\mathbb{P}[Y > X]$. This will happen if the biased and fair coins are (T, H) or (T, T) which happens with probability:

$$\frac{2}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}!$$

So, two thirds of the time Y is larger than X even though X is larger on average! The source of this apparent confusion is that while the scores X and Y can take on different values, the *binary* variable W, whether Y is larger than X, is equal to one or zero, and so:

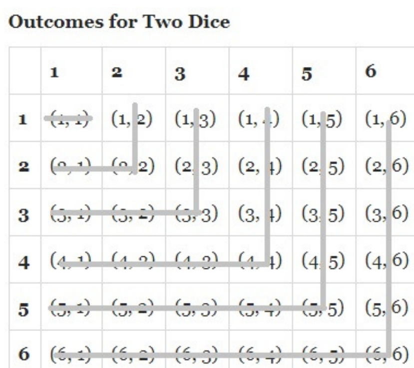
$$\mathbb{E}[W] = 1 \cdot \mathbb{P}[W = 1] + 0 \cdot \mathbb{P}[W = 0] = \mathbb{P}[W = 1] = \mathbb{P}[Y > X],$$

so it is also an expectation.

The intention was that students would calculate the expectations of X and Y so this is the calculation we will present here. The outcomes for X are $\sqrt{\frac{1}{6}}, \sqrt{\frac{2}{6}}, \ldots, \sqrt{\frac{6}{6}}$, each occurring with probability 1/6. It follows that:

$$\mathbb{E}[X] = \sum_i X_i \mathbb{P}[X = X_i] = \sqrt{\frac{1}{6}} \cdot \frac{1}{6} + \sqrt{\frac{2}{6}} \cdot \frac{1}{6} + \cdots \sqrt{\frac{6}{6}} \cdot \frac{1}{6} \approx 0.7370.$$

The difficulty is with Bob. But note that we can 'lump' together his scores using the outcome space:



Outcomes for Two Dice

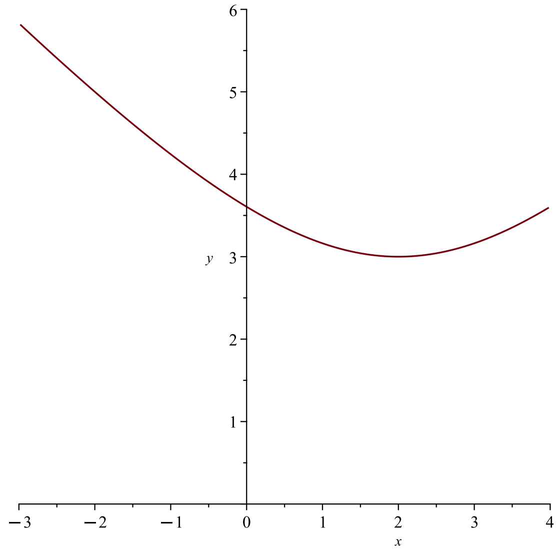
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

On each of these 'lines', the maximum of the two rolls are equal, growing $1, 2, \ldots, 6$, each giving scores $1/6$, $2/6, \ldots, 6/6$, with growing probabilities $1/36$, $3/36, \ldots, 11/36$ giving:

$$\mathbb{E}[Y] = \sum_i X_i \mathbb{P}[X = X_i] = \frac{1}{6} \cdot \frac{1}{36} + \frac{2}{6} \cdot \frac{3}{36} + \cdots \frac{6}{6} \cdot \frac{11}{36} = \frac{161}{216} \approx 0.7454,$$

so Bob's expected score is higher.

7. Consider the curve $y = \sqrt{x^2 - 4x + 13}$. Find the point on the curve that closest to the origin.



Solution: Let $P(x, y)$ be a point on the curve, so that $P(x, \sqrt{x^2 - 4x + 13})$, and $O(0, 0)$ the origin. We want to minimise

$$d(x, y) = |OP| = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2} = \sqrt{x^2 + x^2 - 4x + 13} = \sqrt{2x^2 - 4x + 13}.$$

To reduce this to a more elementary problem, note that on positive numbers, squaring is increasing:

$$0 \leq a < b \underset{\text{sq}}{\Longrightarrow} 0 \leq a^2 < b^2,$$

and it follows that minimising $d(x, y) \geq 0$ is equivalent to minimising:

$$d(x, y)^2 = 2x^2 - 4x + 13.$$

This is a '$\bigcup$'-quadratic, whose minimum can be found using various methods including differentiation:

$$\frac{d}{dx}(2x^2 - 4x + 13) = 4x - 4 \overset{!}{=} 0$$
$$\underset{+4}{\Longrightarrow} 4x = 4$$
$$\underset{\div 4}{\Longrightarrow} x = 1$$

It remains to substitute $x = 1$ to find that the point closest to the origin is:

$$P_{\min} = (1, \sqrt{10}).$$

8. Let $a \in \mathbb{R}$. Show that f does *not* have a local minimum at $x = 1$:

$$f(x) = x^4 - 2x^3 + ax^2 + 4x + 1.$$

Solution: This is a basic test of knowing that:

$$f \text{ has a local min at } x = x_0 \implies f'(x_0) = 0 \text{ and } f''(x_0) \geq 0.$$

Suppose first that $f'(1) = 0$:

$$\begin{aligned} f'(x) &= 4x^3 - 6x^2 + 2ax + 4 \\ \implies f'(1) &= 4 - 6 + 2a + 4 = 2 + 2a \overset{!}{=} 0 \\ \implies a &= -1 \\ \implies f'(x) &= 4x^3 - 6x^2 - 2x + 4 \end{aligned}$$

Now consider $f''(x)$:

$$\begin{aligned} f''(x) &= 12x^2 - 12x - 2 \\ \implies f''(1) &= 12 - 12 - 2 < 0 \end{aligned}$$

which implies that f does not have a local minimum at $x = 1$ but in fact a local *maximum*.

9. Suppose that a professor gave a test to four students — Alice, Bob, Carol, and Dan — and wants to let them grade each other's tests. Of course, no student should grade their own test. How many ways could the professor hand the tests back to the students for grading, such that no student received their own test back?

Solution: This involves the interesting mathematics of *derangements*, but just as easy to count by brute force. Let the string $XYZW$ symbolise that Alice, A, corrects X; Bob, B, corrects Y, and so on. We count:

BADC	CADB	DCAB
BCDA	CBDA	DCBA
BDAC	CDAB	DABC

The answer is nine.

10. An ancient Egyptian papyrus, dating from about 2000 BC and discovered only in 1853, contained the following mathematical problem. One hundred sacks of grain is to be divided between five workers in such a way that worker 2 gets more than worker 1 by the same amount as:

- worker 3 gets in excess of worker 2,
- worker 4 gets in excess of worker 3,
- worker 5 gets in excess of worker 4.

In addition, worker 1 and worker 2 together must get the amount that is seven times smaller than the amount the three remaining workers get together. How many sacks of grain did each worker get?

Solution: Let w_i be the number of sacks that worker i gets. We have a number of equations:

$$\begin{aligned} w_1 + w_2 + w_3 + w_4 + w_5 &= 100 \\ w_2 - w_1 &= w_3 - w_2 \\ w_2 - w_1 &= w_4 - w_3 \\ w_2 - w_1 &= w_5 - w_4 \\ w_1 + w_2 & \frac{w_3 + w_4 + w_5}{7} \end{aligned}$$

The good news is that we have five equations in five unknowns, so potentially a unique solution. The problem is that solving systems of this size requires methods beyond Ordinary Level Leaving Cert mathematics... however the middle three equations imply that the w_i are in arithmetic sequence, because if $w_2 - w_1 := d$, then we have a common difference $w_{i+1} - w_i =: d$ with first term $a := w_1$. This simplifies things greatly. First of all, we have formulae $w_i = a + (i-1)d$, and using the sum of an arithmetic sequence, or just kind of manually, we have:

$$\sum_i w_i = 5a + 10d \overset{!}{=} 100.$$

We can also rewrite the final equation:

$$a + a + d = \frac{a + 2d + a + 3d + a + 4d}{7} \implies 11a - 2d = 0,$$

so we have simultaneous equations in two unknowns:

$$\begin{aligned} 5a + 10d &= 100 \\ 11a - 2d &= 0 \\ \implies 55a - 10d &= 0 \\ \implies 60a &= 100 \\ \underset{\div 60}{\implies} a &= \frac{5}{3} \\ 2d &= 11a \\ \implies d &= \frac{55}{6}, \end{aligned}$$

which implies that:

$$w_1 = \frac{5}{3},\ w_2 = \frac{65}{6},\ w_3 = 20,\ w_4 = \frac{175}{6},\ w_5 = \frac{115}{3}.$$